

CLASSIFICATION OF GRADED LEFT-SYMMETRIC ALGEBRA STRUCTURES ON WITT AND VIRASORO ALGEBRAS

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ABSTRACT. We find that a compatible graded left-symmetric algebra structure on the Witt algebra induces an indecomposable module of the Witt algebra with 1-dimensional weight spaces by its left multiplication operators. From the classification of such modules of the Witt algebra, the compatible graded left-symmetric algebra structures on the Witt algebra are classified. All of them are simple and they include the examples given by Chapoton and Kupersmidt. Furthermore, we classify the central extensions of these graded left-symmetric algebras which give the compatible graded left-symmetric algebra structures on the Virasoro algebra. They coincide with the examples given by Kupersmidt.

1. INTRODUCTION

Left-symmetric algebras (or under other names like pre-Lie algebras, quasi-associative algebras, Vinberg algebras and so on) are Lie-admissible algebras (see Proposition 2.2). They were introduced by A. Cayley in 1896 as a kind of rooted tree algebras ([Ca]). They also arose from the study of convex homogenous cones ([V]), affine manifolds and affine structures on Lie groups ([Ko]), deformation of associative algebras ([G]) in 1960s. As it was pointed out in [CL], the left-symmetric algebra “deserves more attention than it has been given”. They appear in many fields of mathematics and mathematical physics. In [Bu2], there is a survey of certain different fields where left-symmetric algebras play an important role, such as vector fields, rooted tree algebras, words in two letters, vertex algebras, operad theory, deformation complexes of algebras, convex homogeneous cones, affine manifolds, left-invariant affine structures on Lie groups. In addition, left-symmetric algebras have close relations with symplectic and complex structures on Lie groups and Lie algebras ([Chu], [LM], [DaM1-2], [AS]), phase spaces of Lie algebras ([Ku1-2], [Ba]), certain integrable systems ([Bo], [SS]), classical and quantum Yang-Baxter equations ([Ku3], [ES], [GS], [DiM]), combinatorics ([E]) and so on. In particular, they play a crucial role in the Hopf algebraic approach of Connes and Kreimer to renormalization of perturbative quantum field theory ([CK]).

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It is not easy to study left-symmetric algebras. Due to the nonassociativity, there is neither a suitable representation theory nor a complete structure theory like other classical algebras such as associative algebras and Lie algebras. For example, it is far from the classification of semisimple left-symmetric algebras (a left-symmetric algebra is called simple if it has not nontrivial ideals and a semisimple left-symmetric algebra is a direct sum of simple ones). In fact, the classification of complex simple left-symmetric algebras are only in low dimensions and some special cases in certain higher dimensions ([Bu1]).

On the other hand, an important approach to study left-symmetric algebras is through the representation theory of their sub-adjacent Lie algebras. It is known that there exists a compatible left-symmetric structure on a Lie algebra \mathcal{G} if and only if \mathcal{G} has an étale affine representation or equivalently, \mathcal{G} has a bijective 1-cocycle associated to a representation (cf. [Me], [Ki]). Unfortunately, the sub-adjacent Lie algebra of a finite-dimensional left-symmetric algebra over a field of characteristic zero cannot be semisimple. So the beautiful representation theory of finite-dimensional semisimple Lie algebras cannot be used here. However, in the case of the infinite dimensional, there exists a semisimple Lie algebra with a compatible left-symmetric algebra structure. One of such examples is given as a left-symmetric Witt algebra, which is regarded as the first important example of infinite-dimensional left-symmetric algebras (it was also regarded as one of the origins of left-symmetric algebras in [Bu2]). Let U be an associative commutative algebra, and $\mathcal{D} = \{\partial_1, \dots, \partial_n\}$ be a system of commuting derivations of U . Then the vector space

$$\text{Vec}(n) = \left\{ \sum_{i=1}^n u_i \partial_i \mid u_i \in U, \partial_i \in \mathcal{D} \right\}, \quad (1.1)$$

is called a left-symmetric Witt algebra under the multiplication ([Ku1], [DL], [Bu2])

$$u \partial_i \circ v \partial_j = u \partial_i(v) \partial_j. \quad (1.2)$$

In particular, when $n = 1$, $U = \mathbb{F}[x, x^{-1}]$ and $\partial = \frac{\partial}{\partial x}$, the $W^1 = \text{Vec}(1)$ is a simple left-symmetric algebra, whose sub-adjacent Lie algebra is the Witt algebra.

Therefore, it is natural to consider the classification of the compatible left-symmetric algebra structures on the Witt algebra. It is easy to know that all compatible left-symmetric algebras on the Witt algebra are simple. Since the Witt algebra is graded, it is also natural to suppose that the compatible left-symmetric algebras should be graded. Hence, in this paper, we mainly consider the left-symmetric algebras with a basis $\{x_n \mid n \in \mathbb{Z}\}$ satisfying

$$x_i x_j = f(i, j) x_{i+j}, \quad [x_i, x_j] = x_i x_j - x_j x_i = (j - i) x_{i+j}, \quad (1.3)$$

where f is a complex-value function on $\mathbb{Z} \times \mathbb{Z}$. In particular, W^1 is just the case $f(i, j) = 1 + j$ and it is a Novikov algebra, which is a left-symmetric algebra with commutative right multiplication operators ([GD], [BN]).

Moreover, like W^1 , some other special cases satisfying equation (1.3) have been already discussed. For example, Chapoton in [Cha] gave the classification in the case $f(i, j) = g(i)h(j)$ (up to a change of basis). As it was pointed out in [Cha], “this Ansatz for the product has no special meaning, except that it allows for a full classification”. Kupershmidt gave a solution $f(i, j) = \frac{j(1+bj)}{1+b(i+j)}$ for $b = 0$ or $b^{-1} \notin \mathbb{Z}$ in [Ku2]. Osborn classified a class of simple infinite dimensional Novikov algebras which includes a classification of Novikov algebras satisfying equation (1.3), that is, $f(i, j) = \alpha + j$ for any $\operatorname{Re} \alpha > 0$ or $\operatorname{Re} \alpha = 0, \operatorname{Im} \alpha \geq 0$ ([O]). However, to our knowledge, a complete classification of left-symmetric algebras satisfying equation (1.3) is still unknown.

In this paper, we will mainly use the representation theory of the Witt algebra. We find that a regular representation of the Witt algebra induced by a compatible left-symmetric algebra structure satisfying equation (1.3) (through its left multiplication operators) belongs to a class of important modules of the Witt algebra, which have been classified. Hence such left-symmetric algebras can be classified through studying the relations between them. As it was done in Lie algebras, it is natural to consider the central extensions of the left-symmetric algebras satisfying equation (1.3), by which we can get the compatible left-symmetric algebra structures on the Virasoro algebra. It is interesting to see that there exist only one class of non-trivial central extensions which coincides with a result in [Ku2].

We would like to point out some interesting remarks from the following aspects.

- Our methods in this paper provide an approach to the study of the possible compatible left-symmetric algebra structures on the infinite-dimensional (semi)simple Lie algebras with a good representation theory.
- The left-symmetric algebras obtained in this paper are the simple graded left-symmetric algebras of growth one. It is difficult to classify all of it. Our study can be regarded as the first step, as Mathieu solved the analogous problem for Lie algebras ([Ma1-3]).
- It will be also interesting to consider their relations with the vertex (operator) algebras which are the fundamental algebraic structures in conformal field theory, since a vertex algebra is equivalent to a left-symmetric algebra and a Lie conformal algebra with some compatible conditions ([BK], [LL]).

This paper is organized as follows. In Section 2, we give some necessary definitions, notations and basic results on left-symmetric algebras and the representation theory of the Witt and Virasoro algebras. In Section 3, we prove that a left-symmetric algebra structure satisfying equation (1.3) induces an indecomposable module of the Witt algebra with 1-dimensional weight spaces by its left multiplication operators. Therefore, such left-symmetric algebras are classified through the classification of those modules. In Section 4, the non-trivial central extensions of the left-symmetric algebras obtained in Section 3 are discussed. A classification of the compatible left-symmetric algebra structures on the Virasoro algebras are obtained.

Throughout this paper, all algebras are over the complex field \mathbb{C} and the indices $m, n, l, i, j, k \in \mathbb{Z}$, unless otherwise stated.

2. PRELIMINARIES AND FUNDAMENTAL RESULTS

Definition 2.1. Let A be a vector space over a field \mathbb{F} equipped with a bilinear product $(x, y) \rightarrow xy$. A is called a left-symmetric algebra if for any $x, y, z \in A$, the associator

$$(x, y, z) = (xy)z - x(yz) \quad (2.1)$$

is symmetric in x, y , that is,

$$(x, y, z) = (y, x, z), \text{ or equivalently } (xy)z - x(yz) = (yx)z - y(xz). \quad (2.2)$$

Left-symmetric algebras are Lie-admissible algebras (cf. [Me]).

Proposition 2.2. Let A be a left-symmetric algebra. For any $x \in A$, let L_x denote the left multiplication operator, that is, $L_x(y) = xy$ for any $y \in A$. Then we have the following results:

(1) The commutator

$$[x, y] = xy - yx, \quad \forall x, y \in A, \quad (2.3)$$

defines a Lie algebra $\mathcal{G}(A)$, which is called a sub-adjacent Lie algebra of A and A is also called a compatible left-symmetric algebra structure on the Lie algebra $\mathcal{G}(A)$.

(2) Let $L : \mathcal{G}(A) \rightarrow gl(A)$ with $x \mapsto L_x$. Then (L, A) gives a representation of the Lie algebra $\mathcal{G}(A)$, that is,

$$[L_x, L_y] = L_{[x, y]}, \quad \forall x, y \in A. \quad (2.4)$$

We call it a regular representation of the Lie algebra $\mathcal{G}(A)$.

There is not a compatible left-symmetric algebra structure on any Lie algebra \mathcal{G} . A sufficient and necessary condition for a Lie algebra with a compatible left-symmetric algebra structure is

given as follows. Let \mathcal{G} be a Lie algebra and $\rho : \mathcal{G} \rightarrow gl(V)$ be a representation of \mathcal{G} . A 1-cocycle $q : \mathcal{G} \rightarrow V$, the linear map on vector space associated to ρ (denoted by (ρ, q)) satisfying

$$q[x, y] = \rho(x)q(y) - \rho(y)q(x), \quad \forall x, y \in \mathcal{G}. \quad (2.5)$$

Let A be a left-symmetric algebra and $\rho : \mathcal{G}(A) \rightarrow gl(V)$ be a representation of its sub-adjacent Lie algebra. If g is a homomorphism of the representations from A to V , then g is a 1-cocycle of $\mathcal{G}(A)$ associated to ρ .

Proposition 2.3. Let \mathcal{G} be a Lie algebra. Then there is a compatible left-symmetric algebra structure on \mathcal{G} if and only if there exists a bijective 1-cocycle of \mathcal{G} .

In fact, let (ρ, q) be a bijective 1-cocycle of \mathcal{G} , then

$$x * y = q^{-1}\rho(x)q(y), \quad \forall x, y \in \mathcal{G}, \quad (2.6)$$

defines a left-symmetric algebra structure on \mathcal{G} . Conversely, for a left-symmetric algebra A , the identity transformation id is a 1-cocycle of $\mathcal{G}(A)$ associated to the regular representation L .

On the other hand, we recall the definition of the Witt algebra W and Virasoro algebra \mathcal{V} . The Witt algebra W (of rank one) is a complex Lie algebra with a basis $\{x_n \mid n \in \mathbb{Z}\}$ whose commutation relations satisfy

$$[x_m, x_n] = (n - m)x_{m+n}. \quad (2.7)$$

The Virasoro algebra \mathcal{V} is the central extension of W defined by the 2-cocycle

$$\Omega(x_m, x_n) = \frac{1}{12}(n^3 - n)\delta_{m+n,0}.$$

That is, \mathcal{V} is a complex Lie algebra with a basis $\{\theta, x_n \mid n \in \mathbb{Z}\}$ whose commutation relations satisfy

$$[\theta, x_n] = 0, \quad [x_m, x_n] = (n - m)x_{m+n} + \delta_{m+n,0}\frac{n^3 - n}{12}\theta. \quad (2.8)$$

Since an ideal of a left-symmetric algebra is still an ideal of its sub-adjacent Lie algebra and W is a simple Lie algebra, we have the following conclusion.

Proposition 2.4. Any compatible left-symmetric algebra structure on the Witt algebra W is simple.

For any module V of a Lie algebra \mathcal{G} , the action of \mathcal{G} on V is denoted by xv for any $x \in \mathcal{G}$ and $v \in V$. Recall that a module V of a Lie algebra is called indecomposable if V cannot be decomposed into a direct sum of two proper submodules. A weight space of a module V of the Witt algebra W or the Virasoro algebra \mathcal{V} is the non-zero vector space V_λ defined by

$$V_\lambda = \{x \in V \mid x_0v = \lambda v\}. \quad (2.9)$$

For later use, we give the description of the indecomposable modules with 1-dimensional weight spaces of the Virasoro algebra \mathcal{V} ([KS], [MP1-2], [Ma4]).

–The \mathcal{V} -module $A_{\alpha,\beta}$ of Feigin-Fuchs with $\alpha, \beta \in \mathbb{C}$ and $0 \leq \operatorname{Re} \alpha < 1$, whose action on a basis $\{v_n \mid n \in \mathbb{Z}\}$ is given by

$$x_i v_n = (\alpha + n + i\beta) v_{n+i}, \quad \theta v_n = 0.$$

–The maximal proper \mathcal{V} -submodule of $A_{0,1}$, called $A'_{0,1} = A_{0,1} \setminus \mathbb{C}v_0$ ($A_{0,1}/A'_{0,1}$ is trivial and $A_{0,0}/\mathbb{C}v_0 \simeq A'_{0,1}$), whose action on a basis $\{v_n \mid n \in \mathbb{Z} \setminus \{0\}\}$ is given by

$$x_i v_n = (n + i) v_{n+i}, \quad \theta v_n = 0, \quad \forall n \neq 0.$$

– The \mathcal{V} -module A_α with $\alpha \in \mathbb{C}$ whose action on a basis $\{v_n \mid n \in \mathbb{Z}\}$ is given by

$$x_i v_n = (n + i) v_{n+i}, \quad \forall n \neq 0,$$

$$x_i v_0 = i(\alpha + i) v_i, \quad \theta v_n = 0.$$

– The \mathcal{V} -module B_β with $\beta \in \mathbb{C}$ whose action on a basis $\{v_n \mid n \in \mathbb{Z}\}$ is given by

$$x_i v_n = n v_{n+i}, \quad \text{if } n + i \neq 0,$$

$$x_i v_{-i} = -i(\beta + i) v_0, \quad \theta v_n = 0.$$

Theorem 2.5.([KS]) Any indecomposable nontrivial module of the Virasoro algebra \mathcal{V} with 1-dimensional weight spaces is isomorphic to one of $A_{\alpha,\beta}, A'_{0,1}, A_\alpha, B_\beta$.

Obviously, $A_{\alpha,\beta}, A'_{0,1}, A_\alpha$, and B_β are also the modules of the Witt algebra W .

Corollary 2.6. Any indecomposable nontrivial module of the Witt algebra W with 1-dimensional weight spaces is isomorphic to one of $A_{\alpha,\beta}, A'_{0,1}, A_\alpha, B_\beta$.

3. COMPATIBLE LEFT-SYMMETRIC ALGEBRA STRUCTURES ON THE WITT ALGEBRA

As we said in the introduction, we study the following compatible left-symmetric algebra structure on the Witt algebra W with the multiplication

$$x_m x_n = f(m, n) x_{m+n}, \tag{3.1}$$

where $\{x_n \mid n \in \mathbb{Z}\}$ is a basis of W satisfying equation (2.7). To avoid confusion, we denote such a left-symmetric algebra structure by V . Then V is a regular module of W defined by the left multiplication operators of V , that is, in equation (3.1), we let $x_m \in W$ and $x_n, x_{m+n} \in V$.

In fact, V is a compatible left-symmetric algebra structure on W if and only if

$$[x_m, x_n] = x_m x_n - x_n x_m = (n - m) x_{m+n} \text{ and } (x_m, x_n, x_l) = (x_n, x_m, x_l). \tag{3.2}$$

They hold if and only if $f(m, n)$ satisfies the following equations:

$$f(m, n) - f(n, m) = n - m, \quad (3.3)$$

$$(n - m)f(m + n, l) = f(n, l)f(m, n + l) - f(m, l)f(n, m + l). \quad (3.4)$$

Lemma 3.1. $f(m, 0) = f(0, 0)$.

Proof. Let $n = l = 0$ in equation (3.4), we have

$$-mf(m, 0) = f(0, 0)f(m, 0) - f(m, 0)f(0, m).$$

Then by equation (3.3), we have

$$f(m, 0)(f(0, m) - m - f(0, 0)) = f(m, 0)(f(m, 0) - f(0, 0)) = 0.$$

Therefore, $f(m, 0) = 0$ or $f(m, 0) = f(0, 0)$.

If $f(0, 0) = 0$, then $f(m, 0) = 0 = f(0, 0)$. The conclusion holds.

If $f(0, 0) \neq 0$, set

$$I_1 = \{m \in \mathbb{Z} \mid f(m, 0) = 0\}, \quad I_2 = \{m \in \mathbb{Z} \mid f(m, 0) = f(0, 0)\}.$$

Obviously, $I_1 \cup I_2 = \mathbb{Z}$, $I_1 \cap I_2 = \emptyset$ and $0 \in I_2$.

Let $l = 0$ in equation (3.4), we have

$$(n - m)f(m + n, 0) = f(n, 0)f(m, n) - f(m, 0)f(n, m). \quad (*)$$

Then we obtain the following results:

- (a) If $m, n \in I_s, m \neq n$ ($s = 1, 2$), then $m + n \in I_s$;
- (b) If $m \in I_1$, then $-m \in I_2$.

Next we prove that $1, -1, 2, -2 \notin I_1$, that is $1, -1, 2, -2 \in I_2$. Thus by the above result (a), it is easy to know that $I_2 = \mathbb{Z}$ and $I_1 = \emptyset$. The conclusion holds.

(I). $1 \notin I_1$.

Otherwise, we suppose $1 \in I_1$. Hence $-1 \in I_2$, $2 \in I_1$ (if $2 \in I_2$, then $1 = 2 + (-1) \in I_2$ which is a contradiction). Therefore, for any $n > 0$, we know $n \in I_1$ by induction. Thus

$$I_1 = \{1, 2, \dots\}, \quad I_2 = \{0, -1, -2, \dots\}.$$

If $-n \geq m > 0$, then equation (*) becomes

$$(n - m)f(0, 0) = f(0, 0)f(m, n).$$

Hence $f(m, n) = n - m$, $f(n, m) = 0$. In particular, if $n < 0$, we have $f(n, 1) = 0$.

If $m > -n > 0$, then $m + n > 0$, and equation (*) becomes

$$f(0, 0)f(m, n) = 0.$$

Hence $f(m, n) = 0$, $f(n, m) = m - n$. In particular, if $m \geq -n > 0$, $f(n, m + 1) = m + 1 - n$.

Let $l = 1$ in equation (3.4), we have

$$(n - m)f(m + n, 1) = f(n, 1)f(m, n + 1) - f(m, 1)f(n, m + 1).$$

Therefore when $m \geq -n > 0$, the above equation becomes

$$(n - m)f(m + n, 1) = (n - m - 1)f(m, 1).$$

Let $m = -n > 0$. Then we have

$$2mf(0, 1) = (2m + 1)f(m, 1), \quad m \geq 1,$$

where $f(0, 1) = 1$ since $1 \in I_1$ and $f(1, 0) = 0$.

Let $m = 1 - n > 1$. Then we have

$$(2m - 1)f(1, 1) = 2mf(m, 1), \quad m \geq 2.$$

Thus, by the above two equations, we know that

$$f(1, 1) = \frac{4m^2}{4m^2 - 1}$$

for all $m \geq 2$, which is impossible.

(II). $-1 \notin I_1$. The proof is similar to the proof in Case (I) by symmetry.

(III). $2 \notin I_1$.

Otherwise suppose $2 \in I_1$. Therefore $1, 0, -1, -2 \in I_2$. Now for any $n \geq 2$, we have $n \in I_1$ by induction since $n = (n + 1) + (-1)$. Thus

$$I_1 = \{2, 3, \dots\}, \quad I_2 = \{1, 0, -1, -2, \dots\}.$$

If $m - 1 > -n > 0$, that is, $m + n > 1$, $m > 1$, then equation (*) becomes

$$0 = f(0, 0)f(m, n).$$

Hence $f(m, n) = 0$, $f(n, m) = m - n$. In particular, if $m \geq -n \geq 2$, then we have $f(m + 2, n) = 0$, $f(n, m + 2) = m + 2 - n$.

If $-m \geq n > 1$, then equation (*) becomes

$$(n - m)f(0, 0) = -f(0, 0)f(n, m).$$

Hence $f(n, m) = m - n$, $f(m, n) = 0$. In particular, if $n \leq -2$, then $f(n, 2) = 0$.

Let $l = 2$ in equation (3.4), we have

$$(n - m)f(m + n, 2) = f(n, 2)f(m, n + 2) - f(m, 2)f(n, m + 2).$$

Therefore, when $m \geq -n \geq 2$, the above equation becomes

$$(m - n)f(m + n, 2) = (m + 2 - n)f(m, 2).$$

Let $m = -n \geq 2$. Then we have

$$2mf(0, 2) = (2m + 2)f(m, 2), \quad m \geq 2,$$

where $f(0, 2) = 2$ since $2 \in I_1$ and $f(2, 0) = 0$.

Let $m = 1 - n \geq 3$. Then we have

$$(2m - 1)f(1, 2) = (2m + 1)f(m, 2), \quad m \geq 3.$$

Thus, by the above two equations, we know that

$$f(1, 2) = \frac{4m^2 + 2m}{2m^2 + m - 1}$$

for all $m \geq 3$, which is impossible.

Case (IV). $-2 \notin I_1$. The proof is similar to the proof in Case (III) by symmetry. \square

Lemma 3.2. The weight space of V is 1-dimensional.

Proof. By Lemma 3.1 and equation (3.3), we have $f(0, m) = f(0, 0) + m$, that is, the elements $\{x_n\}$ are in different eigenspaces of x_0 . So every weight space of V is 1-dimensional. \square

Lemma 3.3. V is an indecomposable W -module.

Proof. We assume that $V = V_1 \oplus V_2$, where V_1, V_2 are proper submodules of V . We assume that V_2 is indecomposable and $x_0 \notin V_2$ without losing generality. It is well known that any submodule of a weight module is still a weight module ([Ka]). Then there exist two nonempty subsets I_1, I_2 of \mathbb{Z} , such that

$$0 \notin I_2, \quad I_1 \cup I_2 = \mathbb{Z}, \quad I_1 \cap I_2 = \emptyset, \quad \text{and} \quad V_i = \bigoplus_{n \in I_i} \mathbb{C}x_n, \quad i = 1, 2.$$

Let s be the smallest positive integer in I_2 (if all elements of I_2 are negative, then let s be the largest negative integer, and the discussion is similar). We have the following results.

(i) $f(m, 0) = 0, f(0, m) = m$. In fact, $x_s x_0 = f(s, 0)x_s \in V_1 \cap V_2$ since $x_0 \in V_1$. So $f(s, 0) = 0$. Therefore we have $f(m, 0) = f(s, 0) = 0$ by Lemma 3.1.

(ii) If $m, n \in I_2$ and $m \neq n$, then $m + n \in I_2$. In fact, since $f(m, n) - f(n, m) = n - m \neq 0$, at least one of $f(m, n), f(n, m)$ is not zero. Suppose $f(m, n) \neq 0$ without losing generality. Then $x_m x_n = f(m, n)x_{m+n} \neq 0$. Hence $x_{m+n} \in V_2$.

(iii) If $m \in I_2$, then $-m \in I_1$. Otherwise, by (ii), we know that $0 = m + (-m) \in I_2$, which contradicts to our assumption.

(iv) If $m \in I_k$ ($k = 1, 2$), $f(n, m) \neq 0$, then $m + n \in I_k$.

(v) If $m \in I_2$ and $n \in I_1$, then $m + n \in I_1$ if and only if $f(n, m) = 0$, $f(m, n) = n - m$ and $m + n \in I_2$ if and only if $f(m, n) = 0$, $f(n, m) = m - n$.

Both (iv) and (v) immediately follow from $x_n x_m = f(n, m) x_{m+n}$.

(vi) $s + 1 \in I_2$. In fact, let $-m = n = 1$ and $l = s$ in equation (3.4), we have,

$$2f(0, s) = f(1, s)f(-1, s+1) - f(-1, s)f(1, s-1). \quad (**)$$

By (i), $f(0, s) = s$. Since $s - 1 \in I_1$ due to our assumption on s , $f(-1, s) = 0$ by (iv). Therefore we know that $2s = f(1, s)f(-1, s+1) \neq 0$. Hence $f(1, s) \neq 0$. So $s + 1 \in I_2$ by (iv).

If $s = 1$, then $2 \in I_2$. Moreover, $\{n \in \mathbb{Z} \mid n \geq 1\} \subset I_2$ by (ii) and $\{n \in \mathbb{Z} \mid n \leq 0\} \subset I_1$ by (iii). That is,

$$I_1 = \{0, -1, -2, \dots\}, \quad I_2 = \{1, 2, \dots\}.$$

By equation (**) and (v), we know that $f(1, 1) = \frac{2}{3}$. Let $m = -n = 2$ and $l = s = 1$ in equation (3.4), we know that $f(2, 1) = \frac{4}{5}$. Let $m = 2, n = -1, l = s = 1$ in equation (3.4), we have

$$-3f(1, 1) = f(-1, 1)f(2, 0) - f(2, 1)f(-1, 3).$$

Therefore, $2 = \frac{16}{5}$, which is a contradiction.

Now supposing that $s > 1$, we have $x_{s-1} \in V_1, x_1 \in V_1$. Hence $x_1 x_{s-1} \in V_1 \cap V_2$ and $x_{s-1} x_1 \in V_1 \cap V_2$. Therefore $f(1, s-1) = f(s-1, 1) = 0$. Since $f(1, s-1) - f(s-1, 1) = s-2$ by equation (3.3), we know that $s-2 = 0$, that is, $s = 2$. Moreover, $-1 \in I_1$ by (ii), $-2 \in I_1$ by (iii) and $3 \in I_2$ by (vi). By equation (**) and (v), we know that $f(1, 2) = 1$. Let $m = -n = 2, l = s = 2$ in equation (3.4), we have

$$-4f(0, 2) = f(-2, 2)f(2, 0) - f(2, 2)f(-2, 4).$$

By (i), $f(0, 2) = 2$ and $f(2, 0) = 0$. Hence $f(2, 2) \neq 0$. Then $4 = 2+2 \in I_2$ by (iv). Furthermore, by (v), we know that $f(-2, 4) = 6$. Therefore $f(2, 2) = \frac{4}{3}$. Let $m = 2, n = -1, l = s = 2$ in equation (3.4), we have

$$-3f(1, 2) = f(-1, 2)f(2, 1) - f(2, 2)f(-1, 4).$$

Therefore, $3 = \frac{20}{3}$, which is a contradiction, too.

Thus there does not exist such an s and hence V is an indecomposable W -module. \square

Corollary 3.4. As a W -module, V must be isomorphic to one of $A_{\alpha, \beta}$, $A'_{0,1}$, A_α , B_β .

Next we discuss the existence and the classification of the left-symmetric algebras V .

Theorem 3.5. As a W -modules, V is not isomorphic to $A'_{0,1}$.

Proof. Suppose that $V \cong A'_{0,1}$, where V is given by the multiplication $x_m x_n = f(m, n)x_{m+n}$. Then there exists an isomorphism $g : V \rightarrow A'_{0,1}$ of modules such that $g(x_0) = \sum_{i \neq 0} c_i v_i$, for a finite number of nonzero $c_i \in \mathbb{C}$. Since $g(x_0 x_0) = x_0 g(x_0)$, we have

$$\sum_{i \neq 0} f(0, 0) c_i v_i = \sum_{i \neq 0} c_i i v_i.$$

Because g is an isomorphism, there exists $k \neq 0$, such that $c_k \neq 0$ and $f(0, 0) = k$. Then $g(x_0) = c_k v_k$ and by Lemma 3.1, $f(m, 0) = k$. Since $g(x_{-k} x_0) = x_{-k} g(x_0)$, we know that $f(-k, 0) g(x_{-k}) = 0$. Therefore $g(x_{-k}) = 0$, which is a contradiction. \square

By a similar discussion as above, we have the following conclusion.

Theorem 3.6. As a W -modules, V is not isomorphic to A_α .

Theorem 3.7. There are compatible left-symmetric algebra structures $V_{\alpha, \epsilon}$ on the Witt algebra W given by the multiplication

$$x_m x_n = \frac{(\alpha + n + \alpha \epsilon m)(1 + \epsilon n)}{1 + \epsilon(m + n)} x_{m+n}, \quad (3.5)$$

where $\alpha, \epsilon \in \mathbb{C}$ satisfying $\epsilon = 0$ or $\epsilon^{-1} \notin \mathbb{Z}$.

Proof. Suppose that $V \cong A_{\alpha, \beta}$, where $\alpha, \beta \in \mathbb{C}$, $0 \leq \operatorname{Re} \alpha < 1$, and V is given by the multiplication $x_m x_n = f(m, n)x_{m+n}$. Then there exists an isomorphism $g : V \rightarrow A_{\alpha, \beta}$ of modules such that $g(x_0) = \sum_i c_i v_i$ for a finite number of nonzero $c_i \in \mathbb{C}$. Since $g(x_0 x_0) = x_0 g(x_0)$, we have

$$\sum_i f(0, 0) c_i v_i = \sum_i c_i (\alpha + i) v_i.$$

So there exists k such that $c_k \neq 0$ and $f(0, 0) = \alpha + k$. Then $g(x_0) = c_k v_k$ and by Lemma 3.1, $f(m, 0) = \alpha + k$. Since $g(x_m x_0) = x_m g(x_0)$, we know that

$$f(m, 0) g(x_m) = c_k x_m v_k = c_k (\alpha + k + m\beta) v_{m+k}.$$

If $\alpha + k \neq 0$, then $g(x_m) = \frac{\alpha + k + m\beta}{\alpha + k} c_k v_{m+k}$. Moreover, in this case, $\alpha + k + m\beta \neq 0$ since g is an isomorphism. Since $g(x_m x_n) = x_m g(x_n)$, we have

$$f(m, n) g(x_{m+n}) = \frac{\alpha + k + n\beta}{\alpha + k} c_k x_m v_{n+k} = \frac{\alpha + k + n\beta}{\alpha + k} (\alpha + n + k + m\beta) c_k v_{m+n+k}.$$

Therefore,

$$f(m, n) = \frac{(\alpha + k + n + m\beta)(\alpha + k + n\beta)}{\alpha + k + (m + n)\beta}.$$

Replacing $\alpha + k$ and $\frac{\beta}{\alpha+k}$ by α and ϵ respectively, we have

$$f(m, n) = \frac{(\alpha + n + \alpha\epsilon m)(1 + \epsilon n)}{1 + \epsilon(m + n)},$$

where $\alpha \neq 0$ and $\epsilon = 0$ or $\epsilon^{-1} \notin \mathbb{Z}$.

If $\alpha + k = 0$, then $\alpha = k = 0$ since $0 \leq \operatorname{Re} \alpha < 1$. Hence $m\beta c_k v_{m+k} = 0$. Thus $\beta = 0$. Moreover, we have $f(m, 0) = 0$ and $f(0, m) = m$. Since $mg(x_m) = g(x_0 x_m) = x_0 g(x_m)$, we know that $g(x_m) = a_m v_m$, where $a_m \neq 0$. In particular, $a_0 = c_0$. Since

$$f(m, n)a_{m+n}v_{m+n} = f(m, n)g(x_{m+n}) = g(x_m x_n) = x_m g(x_n) = na_n v_{m+n},$$

we know that $f(m, n) = \frac{na_n}{a_{m+n}}$. By equation (3.3), we have

$$(n - m)a_{m+n} = na_n - ma_m.$$

Let $m = -n$, we have $a_n + a_{-n} = 2a_0$. On the other hand, let $m = 2, n = -1$, we have $a_{-1} = 3a_1 - 2a_2$. Therefore $a_2 = 2a_1 - a_0$. By induction, we know that $a_n = na_1 - (n - 1)a_0$. Replacing $\frac{a_1 - a_0}{a_0}$ by ϵ , we have

$$f(m, n) = \frac{na_n}{a_{m+n}} = \frac{n(na_1 - na_0 + a_0)}{(m + n)a_1 - (m + n)a_0 + a_0} = \frac{n(1 + n\epsilon)}{1 + \epsilon(m + n)},$$

where $\epsilon = 0$ or $\epsilon^{-1} \notin \mathbb{Z}$. Moreover, it is just the case $\alpha = 0$ in equation (3.5).

Conversely, it is easy to know that we define a compatible left-symmetric algebra structure on the Witt algebra W by equation (3.5). \square

Theorem 3.8. There are compatible left-symmetric algebra structures $V^{\beta, k}$ on the Witt algebra W given by the multiplications

$$x_m x_n = (n + k)x_{m+n}, \text{ if } m + n + k \neq 0, \quad (3.6)$$

$$x_{-n-k} x_n = \frac{(n + k)(\beta - n - k)}{\beta - k} x_{-k}, \quad (3.7)$$

where $\beta \in \mathbb{C}$ and $k \in \mathbb{Z}$ satisfying $\beta - k \neq 0$.

Proof. Suppose that $V \cong B_\beta$, where $\beta \in \mathbb{C}$, and V is given by the multiplication $x_m x_n = f(m, n)x_{m+n}$. Then there exists an isomorphism $g : V \rightarrow B_\beta$ of modules such that $g(x_0) = \sum_i c_i v_i$ for a finite number of nonzero $c_i \in \mathbb{C}$. Since $g(x_0 x_0) = x_0 g(x_0)$, we have

$$\sum_i f(0, 0)c_i v_i = \sum_{i \neq 0} c_i i v_i.$$

So there exists k such that $c_k \neq 0$ and $f(0, 0) = k$. Then $g(x_0) = c_k v_k$ and by Lemma 3.1, $f(m, 0) = k$.

Case (I). $k \neq 0$. Since $g(x_m x_0) = x_m g(x_0)$, we have

$$f(m, 0)g(x_m) = c_k k v_{m+k}, \text{ if } m+k \neq 0; \quad f(m, 0)g(x_{-k}) = c_k k(\beta - k)v_0.$$

Hence $g(x_m) = c_k v_{m+k}$, when $m \neq -k$ and $g(x_{-k}) = c_k(\beta - k)v_0$. In this case, $\beta - k \neq 0$ since g is an isomorphism. Since $g(x_m x_n) = x_m g(x_n)$, we have

$$f(m, n)g(x_{m+n}) = c_k x_m v_{n+k} \text{ if } n+k \neq 0,$$

$$f(m, -k)g(x_{m-k}) = c_k(\beta - k)x_m v_0 = 0.$$

Case (I-1). $m+n+k \neq 0$. If $n \neq -k$, we have

$$f(m, n)c_k v_{m+n+k} = c_k x_m v_{n+k} = (n+k)c_k v_{m+n+k}.$$

Thus $f(m, n) = n+k$. If $n = -k$, we have $f(m, -k) = 0$. Therefore we know that

$$f(m, n) = n+k, \text{ for all } m, n \in \mathbb{Z} \text{ with } m+n+k \neq 0.$$

Case (I-2). $m+n+k = 0$. If $n \neq -k$, we have

$$f(-k-n, n)c_k(\beta - k)v_0 = c_k x_m v_{n+k} = (n+k)(\beta - n - k)c_k v_0.$$

Then $f(-k-n, n) = \frac{(n+k)(\beta - n - k)}{\beta - k}$. If $n = -k$, we have $f(0, -k) = 0$. Therefore we know that

$$f(m, n) = \frac{(n+k)(\beta - n - k)}{\beta - k}, \text{ for all } m, n \in \mathbb{Z} \text{ with } m+n+k = 0.$$

Case (II). $k = 0$. Then $f(0, m) = m$. Since

$$x_0 g(x_m) = g(x_0 x_m) = f(0, m)g(x_m) = m g(x_m),$$

we know that $g(x_m) = d_m v_m$, where $d_m \neq 0$. In particular, $d_0 = c_0$. Since

$$(n-m)d_{m+n}v_{m+n} = g([x_m, x_n]) = x_m g(x_n) - x_n g(x_m) = d_n x_m v_n - d_m x_n v_m,$$

we have

$$(n-m)d_{m+n} = n d_n - m d_m, \text{ if } m+n \neq 0; \quad (\beta - n)d_n + (\beta + n)d_{-n} = 2d_0.$$

Let $m = 2$ and $n = -1$, we have $d_{-1} = 3d_1 - 2d_2$. Let $m = -2$ and $n = 1$, we have $d_{-2} = 4d_1 - 3d_2$. On the other hand, let $n = 1$, we have $(\beta - 1)d_1 + (\beta + 1)(3d_1 - 2d_2) = 2d_0$. Let $n = 2$, we have $(\beta - 2)d_2 + (\beta + 2)(4d_1 - 3d_2) = 2d_0$. Therefore, we know that

$$d_0 = \beta d_1, \quad d_1 = d_2 = d_{-1} = d_{-2}.$$

By induction, we have $d_n = d_1$ for all $n \neq 0$ and $d_0 = \beta d_1$, where $\beta \neq 0$ since $d_0 \neq 0$. Because

$$f(m, n)g(x_{m+n}) = g(x_m x_n) = x_m g(x_n),$$

with a similar discussion as in Case (I), we know that

$$f(m, n) = \begin{cases} n, & \text{if } m + n \neq 0; \\ \frac{n(\beta - n)}{\beta}, & \text{if } m + n = 0, \text{ where } \beta \neq 0. \end{cases}$$

Obviously, it is just the case $k = 0$ in equations (3.6) and (3.7).

Conversely, it is easy to know that we define a compatible left-symmetric algebra structure on the Witt algebra W by equations (3.6) and (3.7). \square

Lemma 3.9. Let T be an automorphism of the Witt algebra W . Then $T(x_0) = \pm x_0$.

Proof. Suppose $T(x_0) = \sum_i c_i x_i$ and $T(x_m) = \sum_j d_j x_j$ for a finite number of nonzero $c_i, d_j \in \mathbb{C}$. Therefore we have

$$m \sum_j d_j x_j = T([x_0, x_m]) = [T(x_0), T(x_m)] = \sum_{i+j=l} (j-i)c_i d_j x_l.$$

If there exists $i > 0$ such that $c_i \neq 0$, then let n and k be the maximal numbers in the sets $\{i \mid c_i \neq 0\}$ and $\{j \mid d_j \neq 0\}$ respectively. Therefore, by comparing the coefficient of x_{n+k} in both hand sides of the above equation, we have $0 = (k-n)c_n d_k$. So $k = n$. However, since m is arbitrary, we know that $\text{Im}T \subset \bigoplus_{i \leq n} \mathbb{C}x_i$, which is contradictory to the assumption that T is an automorphism of W . By a similar discussion, we can prove that there does not exist $i < 0$ such that $c_i \neq 0$.

Therefore $T(x_0) = c_0 x_0$. Since $m \sum_j d_j x_j = \sum_j j c_0 d_j x_j$, we know that there exists k such that $d_k \neq 0$ and $m = k c_0$ for any $m \neq 0$. Hence $T(x_m) = d_k x_k$. In particular, there exists $m_1 \in \mathbb{Z}$, such that $T(x_{m_1}) = d_1 x_1$. So $c_0 = m_1$. Moreover, since $c_0 | m$ for any $m \in \mathbb{Z}$, we have $c_0 = \pm 1$. \square

Proposition 3.10. Let (V_1, \cdot) and $(V_2, *)$ be two compatible left-symmetric algebras structures on the Witt algebra W defined by

$$x_m \cdot x_n = f_1(m, n)x_{m+n}, \quad \text{and} \quad x_m * x_n = f_2(m, n)x_{m+n}$$

respectively. If V_1 is isomorphic to V_2 as left-symmetric algebras, then

$$f_1(m, n) = f_2(m, n), \quad \text{or} \quad f_1(m, n) = -f_2(-m, -n).$$

Proof. Let T be an isomorphism from V_1 into V_2 , that is, $T(x \cdot y) = T(x) * T(y)$ for all $x, y \in V_1$. Obviously, T is an automorphism of W . By Lemma 3.9, we have $T(x_0) = \pm x_0$.

If $T(x_0) = x_0$, then there exists $a_m \neq 0$ such that $T(x_m) = a_m x_m$ from the proof of Lemma 3.9. Hence it is obvious that $a_m a_n = a_{m+n}$ and $a_0 = 1$. Since

$$f_1(m, n)a_{m+n}x_{m+n} = T(x_m \cdot x_n) = T(x_m) * T(x_n) = f_2(m, n)a_m a_n x_{m+n},$$

we have $f_1(m, n) = f_2(m, n)$.

If $T(x_0) = -x_0$, then there exists $b_m \neq 0$ such that $T(x_m) = b_m x_{-m}$ from the proof of Lemma 3.9. Hence it is easy to know that $b_m b_n = -b_{m+n}$ and $b_0 = -1$. Since

$$f_1(m, n) b_{m+n} x_{-(m+n)} = T(x_m \cdot x_n) = T(x_m) * T(x_n) = f_2(-m, -n) b_m b_n x_{-(m+n)},$$

we have $f_1(m, n) = -f_2(-m, -n)$. \square

Theorem 3.11. Any graded compatible left-symmetric algebra structure on the Witt algebra W satisfying equation (3.1) is isomorphic to one of the following algebras:

$$V_{\alpha, \epsilon}, \quad \alpha, \epsilon \in \mathbb{C} \text{ satisfying } \epsilon = 0 \text{ or } \epsilon^{-1} \notin \mathbb{Z};$$

$$V^{\beta, k}, \quad \beta \in \mathbb{C} \text{ and } k \in \mathbb{Z} \text{ satisfying } \beta \neq k.$$

Moreover, the isomorphisms between them are exactly given as follows,

$$V_{\alpha, \epsilon} \cong V_{-\alpha, -\epsilon}, \quad V_{\alpha, 0} \cong V_{\alpha, 1/\alpha} \text{ with } \alpha \notin \mathbb{Z} \text{ and } V^{\beta, 0} \cong V^{-\beta, 0}.$$

Proof. The first half part immediately follows from Corollary 3.4 and Theorems 3.5-3.8. For the second half part, we assume that $V_1 \cong V_2$, where V_1, V_2 are the compatible left-symmetric algebra structures on W defined by equation (3.1) with $f_1(m, n)$ and $f_2(m, n)$ respectively. Then by Proposition 3.10, we have $f_1(m, n) = f_2(m, n)$ or $f_1(m, n) = -f_2(-m, -n)$. Obviously, in the former case, $V_1 = V_2$ and in the latter case, $V_1 \cong V_2$ by the linear isomorphism $x_n \rightarrow -x_{-n}$.

Case (I). $V_1 = V_{\alpha_1, \epsilon_1}$ and $V_2 = V_{\alpha_2, \epsilon_2}$.

If $f_1(m, n) = f_2(m, n)$, then $f_1(0, 0) = \alpha_1 = f_2(0, 0) = \alpha_2$. Set $\alpha_1 = \alpha_2 = \alpha$. Since

$$f_1(m, n) = \frac{(\alpha + n + \alpha \epsilon_1 m)(1 + \epsilon_1 n)}{1 + \epsilon_1(m + n)} = \frac{(\alpha + n + \alpha \epsilon_2 m)(1 + \epsilon_2 n)}{1 + \epsilon_2(m + n)} = f_2(m, n),$$

we have

$$(\epsilon_1 - \epsilon_2)[\alpha(\epsilon_1 + \epsilon_2) - 1 + \alpha \epsilon_1 \epsilon_2(m + n)] = 0.$$

If $\alpha = 0$, then $\epsilon_1 = \epsilon_2$. In this case, $V_1 = V_2 = V_{0, \epsilon}$. If $\alpha \neq 0$, then $\epsilon_1 = \epsilon_2$ or $\epsilon_1 = 0, \epsilon_2 = \frac{1}{\alpha}$ or $\epsilon_2 = 0, \epsilon_1 = \frac{1}{\alpha}$. Therefore, we have $V_1 = V_2 = V_{\alpha, \epsilon_1}$, or

$$V_1 = V_{\alpha, 0}, \quad V_2 = V_{\alpha, 1/\alpha}, \alpha \notin \mathbb{Z}; \text{ or } V_1 = V_{\alpha, 1/\alpha}, V_2 = V_{\alpha, 0}, \quad \alpha \notin \mathbb{Z}.$$

If $f_1(m, n) = -f_2(-m, -n)$, then $f_1(0, 0) = \alpha_1 = -f_2(0, 0) = -\alpha_2$. Set $\alpha_1 = -\alpha_2 = \alpha$. Since

$$f_1(m, n) = \frac{(\alpha + n + \alpha \epsilon_1 m)(1 + \epsilon_1 n)}{1 + \epsilon_1(m + n)} = \frac{(\alpha + n - \alpha \epsilon_2 m)(1 - \epsilon_2 n)}{1 - \epsilon_2(m + n)} = -f_2(-m, -n),$$

we have

$$(\epsilon_1 + \epsilon_2)[\alpha(\epsilon_1 - \epsilon_2) - 1 - \alpha \epsilon_1 \epsilon_2(m + n)] = 0.$$

If $\alpha = 0$, then $\epsilon_1 = -\epsilon_2$. In this case, $V_1 = V_{0,\epsilon} \cong V_2 = V_{0,-\epsilon}$. If $\alpha \neq 0$, then $\epsilon_1 = -\epsilon_2$ or $\epsilon_1 = 0$, $\epsilon_2 = -\frac{1}{\alpha}$ or $\epsilon_2 = 0$, $\epsilon_1 = \frac{1}{\alpha}$. Therefore, we have

$$V_1 = V_{\alpha,\epsilon}, V_2 = V_{-\alpha,-\epsilon}; \text{ or } V_1 = V_{\alpha,0}, V_2 = V_{-\alpha,-1/\alpha}, \alpha \notin \mathbb{Z};$$

$$\text{or } V_1 = V_{\alpha,1/\alpha}, V_2 = V_{-\alpha,0}, \alpha \notin \mathbb{Z}.$$

Case (II). $V_1 = V^{\beta_1, k_1}$ and $V_2 = V^{\beta_2, k_2}$.

If $f_1(m, n) = f_2(m, n)$, then $f_1(0, 0) = k_1 = f_2(0, 0) = k_2$. Set $k_1 = k_2 = k$. Since

$$f_1(-n - k, n) = \frac{(n + k)(\beta_1 - n - k)}{\beta_1 - k} = \frac{(n + k)(\beta_2 - n - k)}{\beta_2 - k} = f_2(-n - k, n),$$

we know that $\beta_1 = \beta_2$.

If $f_1(m, n) = -f_2(-m, -n)$, then $f_1(0, 0) = k_1 = -f_2(0, 0) = -k_2$. Set $k_1 = -k_2 = k$. If $k \neq 0$, then let $n \neq 0, -k$. Since $f_1(-n - k, n) = -f_2(n + k, -n)$, we have

$$\frac{(n + k)(\beta_1 - n - k)}{\beta_1 - k} = -(-n - k).$$

So $\beta_1 - n - k = \beta_1 - k$, which is a contradiction. Therefore $k = 0$. Since

$$f_1(-n, n) = \frac{n(\beta_1 - n)}{\beta_1} = -\frac{-n(\beta_2 + n)}{\beta_2} = -f_2(n, -n),$$

we have $\beta_1 = -\beta_2$. On the other hand, when $\beta_1 = -\beta_2 = \beta$ and $k_1 = k_2 = 0$, it is easy to know that $f_1(m, n) = -f_2(-m, -n)$ and then $V_1 = V^{\beta, 0} \cong V_2 = V^{-\beta, 0}$.

Case (III). $V_1 = V_{\alpha,\epsilon}$ and $V_2 = V^{\beta,k}$.

If $f_1(m, n) = f_2(m, n)$, then $f_1(0, 0) = \alpha = f_2(0, 0) = k \in \mathbb{Z}$. Obviously there exist m, n such that $m \neq 0, n \neq 0$ and $m + n + k \neq 0$. Since

$$f_1(m, n) = \frac{(k + n + k\epsilon m)(1 + \epsilon n)}{1 + \epsilon(m + n)} = n + k = f_2(m, n),$$

we have

$$\epsilon mn(k\epsilon - 1) = 0.$$

Since $\epsilon^{-1} \notin \mathbb{Z}$, we know $k\epsilon - 1 \neq 0$. Then $\epsilon = 0$. Let $n \neq 0, -k$. Since

$$f_1(-n - k, n) = k + n = \frac{(k + n)(\beta - k - n)}{\beta - k} = f_2(-n - k, n),$$

we have $\beta - k - n = \beta - k$, which is a contradiction.

If $f_1(m, n) = -f_2(-m, -n)$, then we still have $f_1(m, n) = f_2(m, n)$ by taking $V_1 = V_{-\alpha,-\epsilon}$ and $V_2 = V^{\beta,k}$. Since $V_{\alpha,\epsilon} \cong V_{-\alpha,-\epsilon}$, we know that $V_{\alpha,\epsilon}$ is not isomorphic to $V^{\beta,k}$ in this subcase.

So there does not exist an isomorphism between $V_{\alpha,\epsilon}$ and $V^{\beta,k}$. \square

Example 3.12. In [Cha], the notion of pre-Lie algebra was used. Chapoton classified the simple graded left-symmetric algebras of growth one satisfying the following two conditions:

- (1) The underlying graded vector space is $E = \oplus_{i \in \mathbb{Z}} \mathbb{C}e_i$;
- (2) The product is given by $e_i \circ e_j = f(i)g(j)e_{i+j}$.

Such a left-symmetric algebra is isomorphic either to the algebra A_a defined by

$$e_i \circ e_j = (1 + aj)e_{i+j}, \quad a \in \mathbb{C},$$

or to the algebra B_b defined by

$$e_i \circ e_j = \frac{j}{1 + bi} e_{i+j}, \quad b = 0 \text{ or } b^{-1} \notin \mathbb{Z}.$$

Obviously, A_0 is a commutative associative algebra and $B_0 \cong V_{0,0}$. Moreover, when $a \neq 0$, $A_a \cong V_{\frac{1}{a},0}$ by a linear transformation $e_i \rightarrow \frac{1}{a}e_i$. When $b \neq 0, b^{-1} \notin \mathbb{Z}$, $B_b \cong V_{0,b}$ by a linear transformation $e_i \rightarrow (1+bi)e_i$. Therefore, the only isomorphisms among A_a and B_b are $A_a \cong A_{-a}$ and $B_b \cong B_{-b}$ due to Theorem 3.11, which were given in [Cha], too. Hence, except A_0 , we have obtained all left-symmetric algebras in [Cha].

In particular, $V_{0,b}$ for $b = 0$ or $b^{-1} \notin \mathbb{Z}$ was also given in [Ku2] as a special case of the left-symmetric algebras satisfying equation (3.1). \square

Example 3.13. Recall that a Novikov algebra A is a left-symmetric algebra satisfying $(xy)z = (xz)y$ for any $x, y, z \in A$. Novikov algebras have been introduced in connection with Hamiltonian operators in the formal variational calculus ([GD]) and Poisson brackets of hydrodynamic type ([BN]). It is easy to see that a compatible Novikov algebra structure on the Witt algebra satisfying equation (3.1) must be isomorphic to one of $N_\alpha = V_{\alpha,0}$, that is,

$$x_m x_n = (\alpha + n)x_{m+n}, \quad \alpha \in \mathbb{C}.$$

Moreover, the isomorphisms between them are exactly given by $N_\alpha \cong N_{-\alpha}$. It is just the case (ii) appearing in [O]. \square

4. COMPATIBLE LEFT-SYMMETRIC ALGEBRA STRUCTURES ON THE VIRASORO ALGEBRA

In this section, we consider the central extensions of the left-symmetric algebras obtained in Section 3 whose commutator is the Virasoro algebra \mathcal{V} .

Let (A, \cdot) be a left-symmetric algebra and $\omega : A \times A \rightarrow \mathbb{C}$ be a bilinear form. It defines a multiplication on the space $\hat{A} = A \oplus \mathbb{C}\theta$, by the rule

$$(x + a\theta) * (y + b\theta) = x \cdot y + \omega(x, y)\theta, \quad x, y \in A, \quad a, b \in \mathbb{C}. \quad (4.1)$$

Obviously, \hat{A} is a left-symmetric algebra if and only if

$$\omega(x \cdot y, z) - \omega(x, y \cdot z) = \omega(y \cdot x, z) - \omega(y, x \cdot z). \quad (4.2)$$

\widehat{A} is called a central extension of A . Moreover, by construction, the bilinear form

$$\Omega(x, y) = \omega(x, y) - \omega(y, x), \quad \text{where } x, y \in \mathcal{G}(A), \quad (4.3)$$

defines a central extension of its sub-adjacent Lie algebra $\mathcal{G}(A)$.

Let V be a compatible left-symmetric algebra structure on the Witt algebra W satisfying equation (3.1). Since the Virasoro algebra \mathcal{V} is a central extension of the Witt algebra W , it is natural to consider the central extension $\widehat{V} = V \oplus \mathbb{C}\theta$ of V such that \widehat{V} is a compatible left-symmetric algebra structure on the Virasoro algebra \mathcal{V} while θ being the annihilator of \mathcal{V} , that is, the products are given by

$$\theta\theta = x_m\theta = \theta x_m = 0 \text{ and } x_mx_n = f(m, n)x_{m+n} + \omega(x_m, x_n)\theta \quad (4.4)$$

where $f(m, n)$ satisfies equations (3.3)-(3.4).

We denote $\omega(x_m, x_n)$ by $\omega(m, n)$ for convenience. Then by equations (2.8), (4.2) and (4.3), we have the following equations

$$\omega(m, n) - \omega(n, m) = \frac{1}{12}(n^3 - n)\delta_{m+n, 0}, \quad (4.5)$$

$$(n - m)\omega(m + n, l) = \omega(m, n + l)f(n, l) - \omega(n, m + l)f(m, l). \quad (4.6)$$

Theorem 4.1. When $\alpha \neq 0$ or $\epsilon = 0$, there does not exist a central extension of $V_{\alpha, \epsilon}$ satisfying equations (4.5)-(4.6). There is exactly one central extension of $V_{0, \epsilon}$ with $\epsilon \neq 0$ and $\epsilon^{-1} \notin \mathbb{Z}$ satisfying equations (4.5)-(4.6), which is given by:

$$\omega(x_m, x_n) = \omega(m, n) = \frac{1}{24}(n^3 - n - (\epsilon - \epsilon^{-1})n^2)\delta_{m+n, 0}. \quad (4.7)$$

Proof. Let $m = -n \neq 0$, $l = 0$ in equation (4.6), we have

$$\omega(0, 0) = \frac{1}{24}\alpha(n^2 - 1).$$

Since $\omega(0, 0)$ does not depend on n , we know that $\alpha = 0$ and $\omega(0, 0) = 0$. Notice that $V_{0, \epsilon}$ is given by (see [Ku2], too)

$$f(m, n) = \frac{n(1 + \epsilon n)}{1 + \epsilon(m + n)}.$$

Let $n = l = 0$ in equations (4.5) and (4.6), then we get

$$\omega(m, 0) = \omega(0, m) = 0.$$

Let $m = 0$ in equation (4.6), thus we know

$$n\omega(n, l) = \omega(0, n + l)f(n, l) - \omega(n, l)f(0, l) = -l\omega(n, l),$$

that is, $(n+l)\omega(n, l) = 0$. So we can assume that

$$\omega(n, l) = \varphi(n)\delta_{n+l,0} \text{ for some map } \varphi : \mathbb{Z} \rightarrow \mathbb{C}.$$

Let $m+n=0$ in equation (4.5), thereby we have

$$\varphi(n) - \varphi(-n) = \frac{1}{12}(-n^3 + n).$$

Let $m+n+l=0$ in equation (4.6), then

$$(n-m)\varphi(m+n) = \varphi(m)f(n, -m-n) - \varphi(n)f(m, -m-n),$$

which gives

$$(n-m)\varphi(m+n) = \frac{(-m-n)(1-\epsilon(m+n))}{1-\epsilon m}\varphi(m) - \frac{(-m-n)(1-\epsilon(m+n))}{1-\epsilon n}\varphi(n).$$

Set $\psi(m) = \frac{\varphi(m)}{m(1-\epsilon m)}$. Then we know that

$$\begin{cases} (1-\epsilon n)\psi(n) + (1+\epsilon n)\psi(-n) = \frac{1}{12}(-n^2 + 1), \\ (n-m)\psi(m+n) = -m\psi(m) + n\psi(n). \end{cases} \quad (***)$$

Let $m=2, n=-1$ in equation $(***)$, so we have

$$(1+\epsilon)\psi(-1) + (1-\epsilon)\psi(1) = 0, \quad -3\psi(1) = -2\psi(2) - \psi(-1).$$

So $\psi(2) = \frac{\epsilon+2}{\epsilon+1}\psi(1)$. Let $m=-2, n=1$ in the first part of equation $(***)$, hence we get

$$3\psi(-1) = 2\psi(-2) + \psi(1).$$

So $\psi(-2) = \frac{\epsilon-2}{\epsilon+1}\psi(1)$. Notice that (by the first part of equation $(***)$)

$$(1-2\epsilon)\psi(2) + (1+2\epsilon)\psi(-2) = -\frac{1}{4}.$$

If $\epsilon = 0$, then

$$\psi(2) + \psi(-2) = 2\psi(1) - 2\psi(1) = 0 = -\frac{1}{4},$$

which is a contradiction. So we suppose that $\epsilon \neq 0$. Then $\psi(1) = \frac{1}{24}(1+\epsilon^{-1})$. Let $n=1$ in the second part of equation $(***)$, we have

$$(1-m)\psi(m+1) = -m\psi(m) + \psi(1).$$

Then

$$(m-1)(\psi(m+1) - \psi(1)) = m(\psi(m) - \psi(1)).$$

Without losing generality, we can assume that $m \geq 2$ (a similar discussion is for $m \leq 0$).

Therefore,

$$\psi(m+1) - \psi(1) = \frac{m}{m-1} \cdot \frac{m-1}{m-2} \cdots \frac{2}{1}(\psi(2) - \psi(1)) = m(\psi(2) - \psi(1)) = \frac{m}{1+\epsilon}\psi(1).$$

Hence

$$\psi(m) = \frac{m-1}{1+\epsilon}\psi(1) + \psi(1) = \frac{1}{24}(1 + \epsilon^{-1}m).$$

So

$$\varphi(m) = \frac{1}{24}m(1 - \epsilon m)(1 + \epsilon^{-1}m),$$

and then

$$\omega(m, n) = \varphi(-n)\delta_{m+n,0} = \frac{1}{24}(n^3 - n - (\epsilon - \epsilon^{-1})n^2)\delta_{m+n,0}.$$

Moreover, it is easy to know that $\omega(m, n)$ satisfies equations (4.5)-(4.6). \square

Theorem 4.2. There does not exist a central extension of $V^{\beta, \epsilon}$ satisfying (4.5)-(4.6).

Proof. Firstly, let $m = -n \neq 0$ and $l = 0$ in equation (4.6), we can get

$$\omega(0, 0) = \frac{1}{24}k(n^2 - 1).$$

Since $\omega(0, 0)$ does not depend on n , we know that $k = 0$ and $\omega(0, 0) = 0$. Let $n = l = 0$ in equations (4.5) and (4.6), we have

$$\omega(m, 0) = \omega(0, m) = 0.$$

On the other hand, let $m = 0$ in equation (4.6), then we have

$$n\omega(n, l) = \omega(0, n+l)f(n, l) - \omega(n, l)f(0, l) = -l\omega(n, l),$$

that is, $(n+l)\omega(n, l) = 0$. So we can assume that

$$\omega(n, l) = \varphi(n)\delta_{n+l,0} \text{ for some map } \varphi : \mathbb{Z} \rightarrow \mathbb{C}.$$

Finally, let $m + n = 0$ in equation (4.5), thus we get

$$\varphi(n) - \varphi(-n) = \frac{1}{12}(-n^3 + n).$$

So $\varphi(1) - \varphi(-1) = 0$. Let $m, n \neq 0, m + n + l = 0$ in equation (4.6), thereby we know

$$(n - m)\varphi(m + n) = (m + n)(\varphi(n) - \varphi(m)).$$

Let $m = 2, n = -1$ and $m = -2, n = 1$ in the above equation respectively, we know that

$$\varphi(2) = \varphi(-2) = 4\varphi(1).$$

Hence $\varphi(2) - \varphi(-2) = 0$. However, on the other hand,

$$\varphi(2) - \varphi(-2) = \frac{1}{12}(-2^3 + 2) = -\frac{1}{2},$$

which is a contradiction. \square

Corollary 4.3. Any compatible left-symmetric algebra structure on the Virasoro algebra \mathcal{V} satisfying equation (4.4) is isomorphic to one of the algebras given by the multiplication

$$x_m x_n = \frac{n(1 + \epsilon n)}{1 + \epsilon(m + n)} x_{m+n} + \frac{\theta}{24} (n^3 - n - (\epsilon - \epsilon^{-1})n^2) \delta_{m+n,0}, \quad (4.8)$$

where θ is an annihilator and $\operatorname{Re} \epsilon > 0, \epsilon^{-1} \notin \mathbb{Z}$ or $\operatorname{Re} \epsilon = 0, \operatorname{Im} \epsilon > 0$.

Remark 4.4. The equation (4.8) was also obtained in [Ku2] as a central extension of the left-symmetric algebra $V_{0,\epsilon}$ for $\epsilon \neq 0, \epsilon^{-1} \notin \mathbb{Z}$, which was given only as a special case of the left-symmetric algebras satisfying equation (3.1). It is interesting to see that in our discussion there does not exist a central extension satisfying equations (4.5)-(4.6) in other cases.

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